Indian Statistical Institute, Bangalore
M. Math.
Second Year, First Semester
Operator Theory

Final Examination — Date : Dec 16, 2020
Maximum marks: 100 — Time: 3 hours
Instructor: B V Rajarama Bhat

(1) Let $C[0,1]$ be the space of complex valued continuous functions on $[0,1]$ with supremum norm.
(i) Define $A : C[0,1] \to C[0,1]$ by
$$Af(x) = f(\frac{x}{2}),\ \forall x \in [0,1], f \in C[0,1].$$
Show that A is not compact.
(ii) Define $B : C[0,1] \to C[0,1]$ by
$$Bf(x) = f(0)x + f(1)x^2,\ \forall x \in [0,1], f \in [0,1].$$
Show that B is compact. [15]

(2) Let $D : C[-1,1] \to C[-1,1]$ be the linear operator defined by
$$Df(x) = \int_{-1}^{1} (x+y)^2 f(y)dy,\ \forall x \in [-1,1], f \in C[-1,1].$$
Show that D is compact. Find its spectrum. [15]

(3) Let $\mathcal{M}$ be the normed linear space of 2×2 complex matrices with operator norm. Define a product on $\mathcal{M}$ by $ab = 0$ for every a, b in $\mathcal{M}$. (i) Show that $\mathcal{M}$ is a non-unital commutative Banach algebra. (ii) Let $\mathcal{M}^+$ be the unitization of $\mathcal{M}$ (with suitable norm). Find the spectrum of the matrix
$$b = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix},$$
as an element of $\mathcal{M}^+$. [15]

(4) Let $\mathcal{A}$ be a unital commutative Banach algebra. Let Δ be the maximal ideal space of $\mathcal{A}$ and let $a \mapsto \hat{a}$ from $\mathcal{A}$ to $C(\Delta)$ be the Gelfand map. Show that the Gelfand map is an isometry, that is, $\|\hat{a}\| = \|a\|$ for all $a \in \mathcal{A}$, if and only if $\|a^2\| = \|a\|^2$ for all $a \in \mathcal{A}$. [15]

(5) Let c be a contraction ($\|c\| \leq 1$) in a unital C^*-algebra. (i) Show that $1 - c^*c$ and $1 - cc^*$ are positive. (ii) Show that $c(1-c^*c)^{\frac{1}{2}} = (1-cc^*)^{\frac{1}{2}}c$. (Hint: First observe $c(1-c^*c)^n = (1-cc^*)^n c$ for all $n \geq 1$). [15]

(6) Let $\mathcal{A}$ be a unital C^*-algebra and let $x \in \mathcal{A}$ be self-adjoint. Suppose $\lambda \in \sigma(x)$. Show that there exists a representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of $\mathcal{A}$ on some Hilbert space $\mathcal{H}$, with a unit vector $v \in \mathcal{H}$ such that
$$\lambda = \langle v, \pi(x)v \rangle.$$
[15]

(7) Fix $n \geq 3$. Let $M_n(\mathbb{C})$ be the C^*-algebra of $n \times n$ matrices. Define a state ϕ on $M_n(\mathbb{C})$ by
$$\phi([x_{ij}]) = \frac{x_{11}}{3} + \frac{2x_{22}}{3},\ [x_{ij}] \in M_n(\mathbb{C}).$$
(i) Find the dimension of the Hilbert space for the minimal GNS construction of ϕ. (ii) Decide as to whether this GNS representation is injective or not (prove your claim). [15]